ESSENTIAL MERIDIONAL SURFACES FOR TUNNEL NUMBER ONE KNOTS

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Abstract. We show that for each pair of positive integers g and n, there are tunnel number one knots, whose exteriors contain an essential meridional surface of genus g, and with $2n$ boundary components. We also show that for each positive integer n, there are tunnel number one knots whose exteriors contain n disjoint, non-parallel, closed incompressible surfaces, each of genus n.

1. Introduction

In this paper we consider essential surfaces, closed or meridional, properly embedded in the exteriors of tunnel number one knots. The exterior of a knot k is denoted by $E(k) = S^3 - int\, N(k)$. Recall that a knot k in S^3 has tunnel number one if there exists an arc τ embedded in S^3 with $k \cap \tau = \partial\tau$, such that $S^3 - int\, N(k \cup \tau)$ is a genus 2 handlebody. Such an arc is called an unknotting tunnel for k. Equivalently, a knot k has tunnel number one if there is an arc τ properly embedded in $E(k)$, such that $E(k) - int\, N(\tau)$ is a genus 2-handlebody; in general, the unknotting tunnels we consider are of this type. Sometimes it is convenient to express a tunnel τ' for a knot k as $\tau' = \tau_1 \cup \tau_2$, where τ_1 is a simple closed curve and τ_2 is an arc connecting τ_1 and $\partial N(k)$; by sliding the tunnel we can pass from one expression to the other.

A surface S properly embedded in a 3-manifold M is essential if it is incompressible, ∂-incompressible, and non-boundary parallel. A surface properly embedded in the exterior of a knot k is meridional if each component of ∂S is a meridian of k. Let M be a compact 3-manifold, and let S be a surface in M, either properly embedded or contained in ∂M. Let k be a knot in the interior of M, intersecting S transversely. Let $\hat{S} = S - int\, N(k)$. The surface $\hat{S}$ is properly embedded in $M - int\, N(k)$, and its boundary on $\partial N(k)$, if any, consists of meridians of k. We say that $\hat{S}$ is meridionally compressible in (M, k), if there is an embedded disk D in M, intersecting k at most once, with $\hat{S} \cap D = \partial D$, so that ∂D is a nontrivial curve on $\hat{S}$, and is not parallel to a component of $\partial\hat{S}$ lying on $\partial N(k)$. Otherwise

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$\hat{S}$ is called meridionally incompressible. In particular if $\hat{S}$ is meridionally incompressible in (M, k), then it is incompressible in $M - k$.

Some results are available on incompressible surfaces in tunnel number one knot exteriors. Regarding meridional surfaces, it is shown in [GR] that the exterior of a tunnel number one knot does not contain any essential meridional planar surface. Another proof of this fact is given in [M]. This says that any tunnel number one knot is indecomposable with respect to tangle sum. Considering closed surfaces, it is shown in [MS] that there are tunnel number one knots whose complements contain an essential torus, and such knots are classified. In [E2] it is proved that for each $g \geq 2$, there exist infinitely many tunnel number one knots whose complements contain a closed incompressible surface of genus g; such surfaces are also meridionally incompressible.

In this paper we prove the following,

Theorem 3.2. *For each pair of integers $g \geq 1$ and $n \geq 1$, there are tunnel number one knots K, such that there is an essential meridional surface S in the exterior of K, of genus g, and with $2n$ boundary components. Furthermore, S is meridionally incompressible.*

This gives a positive answer to question 1.8 in [GR]. It follows from [CGLS] that any of the knots of Theorem 3.2 also contains a closed essential surface of genus ≥ 2. That surface is obtained by somehow tubing the meridional surface. However such a surface will be meridionally compressible.

Combining the construction of [E2,§6] with that of Theorem 3.2, we get the following,

Theorem 3.3. *For each positive integer n, there are tunnel number one knots K, such that in the exterior of K there are n disjoint, non-parallel, closed incompressible surfaces, each of genus n.*

It follows from the construction that one of the surfaces, say S_1, is meridionally compressible while the others are meridionally incompressible. It follows also that the surface S_1 is the closest to $\partial E(K)$, that is, S_1 and $\partial E(K)$ bound a submanifold M which does not contain any of the other surfaces. It follows from [CGLS,2.4.3] that S_1 remains incompressible after performing any non-integral Dehn surgery on K, and then so does any of the other surfaces. This fact, Theorem 3.3 and the observation that the exterior of a tunnel number one knot is a compact 3-manifold with Heegaard genus 2, imply the following.

Corollary. *For each positive integer n, there are closed, irreducible 3-manifolds M, with Heegaard genus 2, such that in M there are n disjoint, non-parallel, closed incompressible surfaces, each of genus n.*

This corollary improves one of the results of [Q], where it is shown that for each n, there exist closed irreducible 3-manifolds with Heegaard genus 2 which contain an incompressible surface of genus n.

In Theorem 3.3 the genus of the surfaces grows as much as the number of surfaces. This fact is essential, i.e., it is not just a consequence of the construction method. It follows from the main Theorem of the recent paper [ES], that it is impossible for an irreducible 3-manifold with Heegaard genus g, with or without boundary, to contain an arbitrarily large number of disjoint and closed incompressible surfaces of bounded genus.

The idea of the proof of Theorem 3.2 is the following: Start with a tunnel number one knot k, and unknotting tunnel τ, and a closed incompressible surface in the complement of k which intersects τ in two points. We know by [MS] and [E2] that such knots do exist. Now take an iterate of k and τ, i.e., a knot k^* formed by the union of two arcs $k^* = k_1 \cup k_2$, where $k_1 = \tau$ and k_2 is an arc lying on $\partial N(k)$. Thus k^* intersects S in two points. It follows that k^* is a tunnel number one knot (see Lemma 3.1); an unknotting tunnel τ^* for k^* is formed by the union of k and an arc joining k to a point in $k_1 \cap k_2$. Slide τ^* so that it becomes an arc with endpoints on k^*, also denoted by τ^*. Now take an iterate of k^* and τ^*; this is a knot k^{**} with tunnel number one which intersects S in as many points as desired. If k^* and k^{**} satisfy certain conditions (Theorem 2.1), the surfaces $S_1 = S - int\, N(k^*)$ and $S_2 = S - int\, N(k^{**})$ are essential meridional surfaces in the exterior of k^* and k^{**}, respectively.

Throughout, 3-manifolds and surfaces are assumed to be compact, connected and orientable. If X is contained in a 3-manifold M, then $N(X)$ denotes a regular neighborhood of X in M; if X is contained in a surface S, then $\eta(X)$ denotes a regular neighborhood of X in S. $\Delta(\alpha, \beta)$ denotes the minimal intersection number of two essential simple closed curves on a torus T.

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2. Construction of essential meridional surfaces

Let k be a knot in S^3, and let $\tau' = \tau_1 \cup \tau_2$ be an unknotting tunnel for k, where τ_1 is a simple closed curve, and τ_2 is an arc with endpoints in $\partial N(k)$ and τ_1. Let S be a closed surface of genus g contained in the exterior of k; then S divides S^3 into two parts, denoted by M_1 and M_2, where, say, k lies in M_2. We say that S is special with respect to k and τ' if it satisfies:

(1) τ_1 is disjoint from S, and τ_2 intersects S transversely in one point, so τ_1 lies in M_1;
(2) S is essential in $E(k)$.

This definition is a variation of the one given in [E2,§6].

Note that by [MS], [E1], there exist knots with these properties when $g = 1$; when $g \geq 2$, the existence of knots like these follows from [E2,6.1]. Note that $M_2 \cap N(\tau_2)$ is a cylinder $R \cong D^2 \times I$, so that $R \cap S$ is a disk $D_1 \cong D^2 \times \{1\}$, and $R \cap N(k)$ is a disk $D_0 \cong D^2 \times \{0\}$. Slide τ_1 over τ_2, to get an arc τ with both endpoints on $D_0 \subset \partial N(k)$, so that $\tau \cap M_2$ consists of two straight arcs contained in R, i.e., arcs which intersect each disk $D^2 \times \{x\}$ transversely in one point. The surface S and the arc τ then intersect in two points. The arc τ has a neighborhood $N(\tau) \cong D^2 \times I$, so that $N(\tau) \cap M_2 \subset R$.

Let P be a solid torus, D_0 a disk contained in ∂P, and $\rho = \{\rho_1, \ldots, \rho_n\}$, a collection of arcs properly embedded in P, so that its endpoints lie in D_0. We say that this forms a toroidal tangle with respect to D_0, and denote it by (P, D_0, ρ).

Recall that the wrapping number of a knot in a solid torus is defined as the minimal number of times that the knot intersects any meridional disk of such solid torus. We define the wrapping number of an arc ρ_i in P as the wrapping number of the knot obtained by

joining the endpoints of ρ_i with an arc in D_0, and then pushing it into the interior of P. This is well defined.

The tangle (P, D_0, ρ) is good if:

(1) Each arc ρ_i has wrapping number ≥ 1 in P, and there is at least one arc ρ_i whose wrapping number in P is ≥ 2;
(2) Each arc ρ_i has no local knots, i.e., if a sphere S intersects ρ_i in two points, then S bounds a ball B such that $B \cap \rho_i$ is an unknotted spanning arc.

If the tangle (P, D_0, ρ) is good, then $D_0 - \partial\rho$ is incompressible in $P - \rho$, i.e., there is no disk D properly embedded in P, disjoint from ρ, with $\partial D \subset D_0$, and such that ∂D is essential in $D_0 - \partial\rho$.

Let A be an annulus in ∂P, essential in P, so that $D_0 \subset A$. The tangle (P, D_0, ρ) is good with respect to A if:

(1) (P, D_0, ρ) is good.
(2) No arc ρ_i is isotopic relative to $\partial\rho_i$ to an arc λ contained in A (ignoring the other arcs).

Let $\hat{k}$ be a knot contained in the interior of $N(k) \cup N(\tau)$. We say that $\hat{k}$ is specially knotted if:

(1) $\hat{k}$ intersects the disk D_0 transversely in $2n$ points, so that $\hat{k} \cap N(\tau)$ consists of n straight arcs in $N(\tau)$ ($\cong D^2 \times I$);
(2) the toroidal tangle $(N(k), D_0, \rho)$ is good, where $\rho = \hat{k} \cap N(k)$;
(3) in the case that k is parallel to a curve lying on S, assume also the following: let γ be the curve on $\partial N(k)$ which cobounds an annulus with a curve on S, and so that γ meets D_0 in one arc. Let $A = \eta(\gamma \cup D_0)$. Then $(N(k), D_0, \rho)$ is good with respect to A,

As $N(\tau) \cap M_2 \subset R$, it follows that $\hat{k} \cap R$ consists of $2n$ straight arcs. So $\hat{k}$ intersects S in $2n$ points. Let $\hat{S} = S \cap E(\hat{k})$. This is a surface properly embedded in $E(\hat{k})$, whose boundary consists of $2n$ meridians of the knot $\hat{k}$.

Theorem 2.1. *Let k be a knot, $\tau' = \tau_1 \cup \tau_2$ an unknotting tunnel for k, and S a surface which is special with respect to k and τ'. Let $\hat{k} \subset N(k) \cup N(\tau)$ be a knot which is specially knotted. Then the surface $\hat{S} = S \cap E(\hat{k})$ is an essential meridional surface in the exterior of $\hat{k}$. Furthermore, if the surface S is meridionally incompressible in (S^3, k), then $\hat{S}$ is meridionally incompressible in $(S^3, \hat{k})$. If S is meridionally compressible, but the wrapping number of some arc ρ_i in $N(k)$ is ≥ 3, where $\rho = N(k) \cap \hat{k}$, then $\hat{S}$ is meridionally incompressible.*

Proof. To prove that the surface $\hat{S}$ is essential in the exterior of $\hat{k}$, it suffices to show that it is incompressible, because any two-sided, connected, incompressible surface in an irreducible 3-manifold with incompressible torus boundary must be ∂-incompressible, unless it is a boundary-parallel annulus, which is not the case here.

Let $S' = (S \cup \partial R - int\,(D_1))$, so S' is isotopic to S, and let $\tilde{S} = S' \cap E(\hat{k})$; then $\tilde{S}$ is a surface isotopic to $\hat{S}$. Denote by M_1' and M_2' the complementary regions of $\tilde{S}$ in $E(\hat{k})$,

where $\partial N(k) \cap E(\hat{k})$ lies in M_2'. Let $T = \partial N(k) - int\,(D_0)$. This is a once punctured torus, which is properly embedded in M_2', i.e., $\tilde{S} \cap T = \partial T = \partial D_0$.

Let D be a compression disk for $\tilde{S}$. Suppose first that it lies in M_1'. As S' is essential in $E(k)$, it follows that ∂D is a trivial curve on S' which bounds a disk $D' \subset S'$, and $D \cup D'$ bounds a 3-ball B. As ∂D is supposed to be essential in $\tilde{S}$, one arc α of $\hat{k}$ contained in M_1' must in fact be contained in the 3-ball B. We may assume that $\alpha = \tau$. Note that ∂D must be isotopic in $S' - \partial\tau$ to ∂D_0. Then the tunnel τ is contained in a 3-ball, which implies that k is the trivial knot. This is a contradiction.

Suppose then that D lies in M_2'. Consider the intersection between T and D. If they do not intersect, then there are two cases: (1) D is contained in $N(k)$. In this case ∂D must lie on D_0, which implies that ∂D is trivial on $\tilde{S}$, or that $D_0 - \rho$ is compressible in $N(k) - \rho$, which contradicts the hypothesis. (2) D is disjoint from $N(k)$. One possibility is that ∂D is isotopic to ∂D_0, but in this case the tunnel τ is, as above, contained in a 3-ball which is impossible. Otherwise, by isotoping D we may assume that ∂D is contained in S, and then D is also a compression disk for S disjoint from $N(k)$, which contradicts the hypothesis that S is incompressible in $E(k)$.

Assume then that D and T have nonempty intersection. This intersection consists of a finite number of arcs and simple closed curves. Assume also that D has been chosen, among all compression disks, to have a minimal number of intersections with T. This implies that any curve or arc of intersection is essential in T, for if one curve (arc) is trivial, then doing surgery on D with the disk bounded by an innermost curve (outermost arc) we get a disk with fewer intersections with T.

Let σ be a simple closed curve of intersection which is innermost in D, so it bounds a disk D' whose interior is disjoint from T. If D' lies in $N(k)$, then σ is either a meridian of T, or it is parallel to ∂T, but in both cases it follows that $D_0 - \rho$ is compressible in $N(k) - \rho$. If the interior of D' is disjoint from $N(k)$, then as k is a nontrivial knot, σ must be trivial on T, which contradicts the choice of D.

Assume then that the intersections between D and T consists only of arcs. Let σ be an outermost arc in D which bounds a disk E. Suppose first that $E \subset N(k)$. Then $\partial E = \sigma \cup \delta$, where $\delta \subset D_0$. It follows that ∂E is nontrivial on $\partial N(k)$, i.e., it is a meridian of $N(k)$, and then each of the ρ_i has wrapping number ≤ 1 in $N(k)$, which contradicts the hypothesis. So E cannot be contained in $N(k)$. Again let $\partial E = \sigma \cup \delta$, where σ is contained in T and δ in $\tilde{S}$. As σ is nontrivial in T then δ is also nontrivial in $\tilde{S} - D_0$. By isotoping D we can ensure that $\delta \cap \partial R$ consists of two arcs; let $E' \subset \partial R$ be a disk containing these arcs in its boundary. Now $E \cup E'$ is an annulus with one boundary component on S, and the other on $\partial N(k)$. Here we apply [CGLS,2.4.3], where M, S, T, r_0 of that theorem correspond in our notation to $M_2 - int\,N(k)$, S, $\partial N(k)$, and the component of $\partial(E \cup E')$ lying on $\partial N(k)$, which we denote also by r_0. Clearly S compresses after performing meridional surgery on $\partial N(k)$. Then part (b) of [CGLS,2.4.3] implies that $\Delta(\mu, r_0) \leq 1$, where μ is a meridian of $\partial N(k)$. So either $\mu = r_0$, or r_0 goes around $\partial N(k)$ once longitudinally. The first possibility implies that S is meridionally compressible, and the second one implies that k is parallel to a curve lying on S. So we are done, unless one of these cases happens. Note that each of these possibilities excludes the other, for if k is parallel to a curve on S,

and S is meridionally compressible, then either S is compressible or S is isotopic to $\partial N(k)$.

Suppose first that S is meridionally compressible. Let σ be an outermost arc in D, which bounds a disk E, so that $\partial E = \sigma \cup \delta$, where σ is contained in T and δ in $\tilde{S}$. As above, there is a disk $E' \subset \partial R$, such that $E \cup E'$ is an annulus with one boundary component on S, and the other is a meridian of $\partial N(k)$. Consider all the outermost arcs on D; by the argument given above we can assume that any one of them determines a curve on T parallel to σ. Let F be a region on D adjacent to one of the outermost arcs, so that all of its intersections with T, except at most one, are outermost arcs. To find such an F, take the collection of arcs in D which are not outermost arcs, and among these choose one which is outermost. $F \subset N(k)$, and then either ∂F is trivial on $\partial N(k)$, or ∂F is a meridian of $N(k)$. ∂F consists of, say, $2m$ consecutive arcs, $\partial F = \sigma_1, \delta_1, \ldots, \sigma_m, \delta_m$, where $\sigma_i \subset T$, and $\delta_i \subset D_0$. Then at least $m-1$ of the arcs are parallel to σ, say $\sigma_1, \ldots, \sigma_{m-1}$. If σ_m is not parallel to σ, then ∂F would go around $N(k)$ once longitudinally, which is impossible, for ∂F bounds a disk in $N(k)$. We conclude that all the arcs σ_i are parallel in T, as in Figure 1.

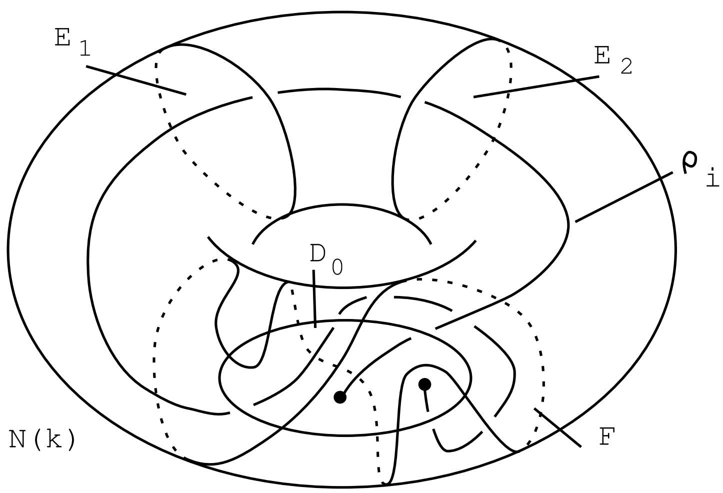

FIGURE 1

Let E_1, E_2 be two meridian disks of $N(k)$ whose boundaries are disjoint from D_0 and $\cup\sigma_i$. Then $E_1 \cup E_2$ bounds a ball B in $N(k)$ which contains D_0 and F, after possibly isotoping F.

There are two cases:

(1) F is parallel to a disk $D_1 \subset \partial N(k)$. Clearly $D_1 \subset B$. F and D_1 cobound a 3-ball B_1. Suppose that an arc ρ_i is contained in B_1. By joining the endpoints of ρ_i with an arc contained in D_0, we get a simple closed curve ρ_i', which is contained in B, and then its wrapping number in $N(k)$ is 0, which contradicts the hypothesis.

So suppose no arc ρ_i is contained in B_1. Consider $D_1 \cap \partial D_0$. This is a collection of arcs which divide D_1 into regions which are in D_0 or in its complement. If there is an outermost arc on D_1 which bounds a disk contained in D_0, then we can isotope F (and

then D) through D_0 to get a compression disk with fewer intersections with T. If no outermost arc bounds a disk lying in D_0, choose any region D_0' of $D_1 \cap D_0$. There is an arc $\alpha \subset D_0'$, whose endpoints lie on ∂D_1 (then $\alpha \subset int\, D_0$), and there is a disk $E_0 \subset B_1$, so that $\partial E_0 = \alpha \cup \beta$, where β is an arc on F. Cut D along E_0, getting two disks; at least one of them is a compression disk for $\tilde{S}$, but it has fewer intersections with T.

(2) ∂F is a meridian of $N(k)$, so ∂F is parallel to ∂E_1 (see Figure 1). So ∂F separates the annulus $\partial B - int(E_1 \cup E_2)$ into two annuli, denoted by A_1 and A_2, where $\partial A_i = \partial E_i \cup \partial F$. Let ρ_i be an arc of ρ, and ρ_i' the simple closed curve obtained by joining the endpoints of ρ_i with an arc in D_0. If the endpoints of ρ_i lie in the same annulus A_j, then ρ_i is isotopic rel $\partial \rho_i$ (when ignoring the other arcs), to an arc disjoint from E_1. This implies that the wrapping number of ρ_i' in $N(k)$ is 0, for ρ_i' is isotopic to a curve disjoint from E_1. If the endpoints of ρ_i lie on different annuli, then ρ_i is isotopic rel $\partial \rho_i$ to an arc which intersects E_1 in one point. This implies that the wrapping number of ρ_i' in $N(k)$ is 1. This contradicts the hypothesis that at least one of the arcs have wrapping number ≥ 2. This completes the proof when the surface S is meridionally compressible.

Suppose now that k is parallel to a curve on S. As before, let σ be an outermost arc in D, which bounds a disk E, so that $\partial E = \sigma \cup \delta$, where σ is contained in T and δ in $\tilde{S}$. Recall that the union of σ and an arc on D_0 is a curve γ on $\partial N(k)$ which cobounds an annulus $E \cup E'$ with a curve on S. Let $A = \eta(\gamma \cup D_0)$. Consider all the outermost arcs on D; recall that any one of them determines a curve on T parallel to σ. Let F be a region on D adjacent to one of the outermost arcs, so that all of its intersections with T, except at most one are outermost arcs. F is then a disk properly embedded in $N(k)$, which intersects D_0 in r arcs, and all the arcs on $T \cap F$, except at most one are parallel. Let $\partial F = \sigma_1 \cup \delta_1 \cup \ldots \sigma_r \cup \delta_r$, where $\sigma_i \subset T$, $\delta_i \subset D_0$, and $\sigma_1, \ldots \sigma_{r-1}$ are parallel to σ. There is an annulus Δ properly embedded in $N(k)$, $\partial \Delta = \partial A$. We can assume that $D_0, \sigma_1, \ldots, \sigma_{r-1}$ are contained in A. If σ_r is not parallel to σ, then it intersects each component of $\partial \Delta$ in one point. It follows that ∂F is trivial in $\partial N(k)$ if and only if each arc σ_i is parallel to σ.

Suppose first that ∂F is trivial in $\partial N(k)$, then $\partial F \subset A$, and F is parallel to a disk $D_1 \subset A$. We can assume that F and Δ do not intersect. F and D_1 cobound a 3-ball B_1. Suppose there is an arc $\rho_i \subset B_1$. The arc ρ_i has no local knots, then it is parallel to an arc $\epsilon_i \subset D_1 \subset A$, i.e., the arc ρ_i is isotopic to an arc lying in A, contradicting the hypothesis. See Figure 2.

If there is no arc ρ_i in B_1, proceed as in the analogous case when $\tilde{S}$ is meridionally compressible, to get a disk $E_0 \subset B_1$, with $\partial E_0 = \alpha \cup \beta$, where $\alpha \subset D_0 \cap D_1$ and $\beta \subset F$, so that by cutting D along E_0, we get another compression disk for $\tilde{S}$ with fewer intersections with T.

Suppose now that ∂F is a meridian of $N(k)$. Then $\partial F = \alpha \cup \beta$, where $\alpha \subset \partial N(k) - A$, $\beta \subset A$, so $\alpha \subset \sigma_r$. The annulus Δ can be isotoped so that $\Delta \cap F$ is a single arc. A and Δ bound a solid torus Δ', and $F \cap \Delta'$ is a meridian disk for Δ'. If ρ_i is any of the arcs of ρ, then ρ_i can be isotoped to be in the 3-ball $\Delta' - int\, N(F)$, and so it is parallel to an arc lying on A. See Figure 3. This completes the proof.

Now we sketch a proof that $\tilde{S}$ is meridionally incompressible. Suppose there is a disk D

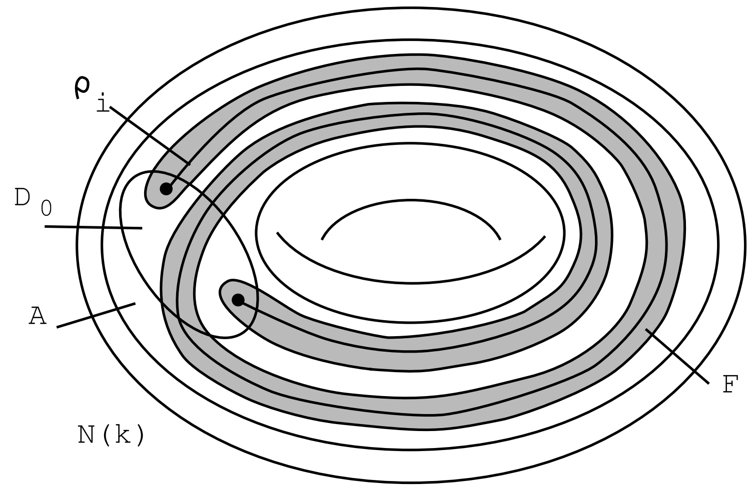

FIGURE 2

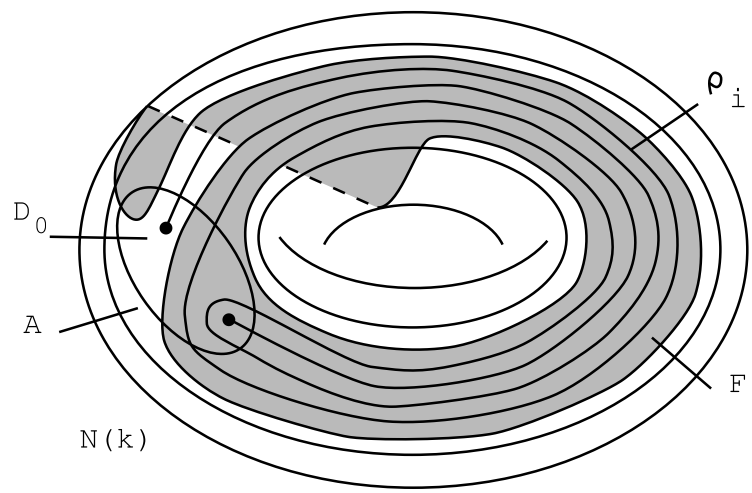

FIGURE 3

embedded in S^3, with $\tilde{S} \cap D = \partial D$, which is a nontrivial curve on $\tilde{S}$, and so that $\hat{k}$ intersects D transversely in one point. If the disk D lies in M_1', then as S' is incompressible in $E(k)$, it follows that ∂D is a trivial curve on S', which is the boundary of a disk contained in S' which intersects $\hat{k}$ once, so D is not a disk of meridional compression.

So assume that D lies in M_2'. Look at the intersections between D and T, and suppose D has been chosen to have minimal intersection with T. This implies that any curve or arc of intersection is essential in T.

Suppose there is a curve of intersection, innermost in D, which bounds a disk D' which meets $\hat{k}$ once. Then D' must lie in $N(k)$. If $\partial D'$ is a meridian of $N(k)$, then each ρ_i has wrapping number ≤ 1. If $\partial D'$ is not a meridian of $N(k)$, then $\partial D'$ bounds a disk in $\partial N(k)$ which either lies in T or contains D_0. In either case it is impossible for ρ to meet D' in

exactly one point. This shows that simple closed curves of intersection cannot bound disks which intersect $\hat{k}$, and then these curves can be removed as before. Suppose there is an outermost arc σ in D which bounds a disk E disjoint from $\hat{k}$. Doing an argument as the one done to prove the incompressibility of $\tilde{S}$, we have that E does not lie in $N(k)$. By the same argument, such a disk can exist only if S is meridionally compressible, or if k is parallel to a curve on S. Note that it is always possible to find an outermost arc which bounds a disk disjoint from $\hat{k}$. So the proof is complete, except if we have one of the cases just mentioned.

Suppose first S is meridionally compressible. In this case we suppose that the wrapping number of some arc ρ_i in $N(k)$ is ≥ 3. Take an outermost arc of intersection in D, and suppose it bounds a disk D' contained in $N(k)$, which intersects $\hat{k}$ in at most one point. $\partial D'$ is a meridian of $N(k)$, and then the wrapping number of any arc ρ_i in $N(k)$ is ≤ 2, contradicting the hypothesis in this case. So suppose all outermost arcs bound disks which do not lie in $N(k)$. As in the proof of the incompressibility of $\tilde{S}$, these arcs in T are all parallel, and each one of them, together with an arc in D_0 is a meridional curve on $\partial N(k)$. If there is a region $F \subset D$, such that all the intersections of F with T, except at most one are outermost arcs, and so that F is disjoint from $\hat{k}$, proceed as in the proof of the incompressibility of $\tilde{S}$. If there is no such region F, then $T \cap D$ consists of m arcs, all of which are outermost arcs in D, so that the complement of the arcs is a single region F' contained in $N(k)$ and intersecting $\hat{k}$ once. There are two cases:

(1) F' is parallel to a disk $D_1 \subset \partial N(k)$. F' and D_1 cobound a 3-ball B_1. If some arc ρ_j is contained in B_1 then its wrapping number in $N(k)$ is 0, contradicting the hypothesis. So there is just one arc ρ_i which intersects B_1; one of its endpoints is in D_1 and the arc intersects F' in one point. As ρ_i has no local knots, $B_1 \cap \rho_i$ is an unknotted spanning arc in B_1. As in the case of the incompressibility, there is a disk $E_0 \subset B_1$, so that $\partial E_0 = \alpha \cup \beta$, where β is an arc on F', and α is an arc in $D_0 \cap D_1$. Cut D along E_0, getting two disks; at least one of them is a meridional compression disk for $\tilde{S}$, but it has fewer intersections with T.

(2) $\partial F'$ is a meridian of $N(k)$. The same proof as in the case of the incompressibility show that if this happens then the wrapping number of any arc ρ_i in $N(k)$ is ≤ 2.

Suppose now that k is parallel to a curve on S. If σ is an outermost arc of intersection in D, bounding a disk E which does not intersect $\hat{k}$, then E is not contained in $N(k)$, and $\partial E = \sigma \cup \delta$, where δ is contained in $\tilde{S}$. The union of σ and an arc on D_0 is a curve γ on $\partial N(k)$ which cobounds an annulus with a curve on S; so γ is a curve which goes around $N(k)$ once longitudinally. If there is an outermost arc of intersection in D bounding a disk D' which intersects $\hat{k}$ in one point, then D' is a meridian disk of $N(k)$. In particular this shows that $D \cap T$ cannot consist of just one arc. As before, if σ' is another outermost arc in D which bounds a disk disjoint from $\hat{k}$, then σ' is parallel to σ.

Now proceed as in the proof of the incompressibility of $\tilde{S}$ in the case that k is parallel to a curve on S. The point is to find a region $F \subset D$, such that all the intersections of F with T, except at most one, are outermost arcs, and so that F is disjoint from $\hat{k}$. If such region exists we are done. If there is an outermost arc on D which bounds a disk which intersects $\hat{k}$, then such region F does exists, for otherwise $D \cap T$ will consist of just one arc. If such

a region F does not exist, the only possibility left is that $D \cap T$ consists of m arcs, all of which are outermost arcs, and $D \cap N(k)$ is a single disk F' which meets $\hat{k}$ once. Then $\partial F'$ is completely contained in the annulus $A = \eta(\gamma \cup D_0)$, which implies that $\partial F'$ is trivial on A. So F' bounds a disk D_1 contained in A; F' and D_1 cobound a ball B_1. If some arc ρ_j is contained in B_1 then it is parallel to an arc lying on A, contradicting the hypothesis. So there is just one arc ρ_i which intersects B_1; one of its endpoints is in D_1 and the arc intersects F' in one point. As ρ_i has no local knots, $B_1 \cap \rho_i$ is an unknotted spanning arc in B_1. As in the proof of the incompressibility of $\tilde{S}$, we can boundary compress D, getting another meridional compression disk for $\tilde{S}$, but with fewer intersections with T. $\square$

Remark. The conditions imposed on the tangle $(N(k), D_0, \rho)$ are somehow local, i.e., they consider each arc separately. Giving to the tangle some global property might produce a slightly stronger theorem.

3. Tunnel number one knots and meridional surfaces.

Let k be a tunnel number one knot, and τ an unknotting tunnel for k which is an embedded arc with endpoints lying on $\partial N(k)$. Assume that a neighborhood $N(k \cup \tau)$ is decomposed as $N(k \cup \tau) = N(k) \cup N(\tau)$, where $N(k)$ is a solid torus, $N(\tau) \cong D^2 \times I$, $N(k) \cap N(\tau)$ consists of two disks E_0 and E_1, and $\tau = \{0\} \times I$.

Let k^* be a knot formed by the union of two arcs, $k^* = k_1 \cup k_2$, such that k_1 is contained in $\partial N(k)$, and $k_2 = \tau$. We say that k^* is an iterate of k and τ.

Lemma 3.1. *Let k and τ be as above, and let k^* be an iterate of k and τ. Then k^* is a tunnel number one knot. An unknotting tunnel β' for k^* is given by the union of k and a straight arc in $N(k)$ connecting k^* and k.*

Proof. $N(k) - k$ is homeomorphic to a product $T \times [0, 1)$. Let δ be a straight arc in $N(k)$ connecting k and one of the points $k_1 \cap k_2$, i.e., it is an arc which intersects each torus $T \times \{x\}$ in one point. Then $\beta' = k \cup \delta$ is an unknotting tunnel for k^*. To see that, slide k_1 over δ and then over k to get a 1-complex which is clearly equivalent to $k \cup \tau$, so its complement is a genus 2 handlebody. $\square$

Let k^* be an iterate of k and τ. It follows by construction that $k^* \subset N(k \cup \tau)$. Also if β' is the unknotting tunnel for k^* given by the lemma, then $k^* \cup \beta' \subset N(k \cup \tau)$. Now β' can be modified to be an arc β with endpoints in k^*. It follows that if k^{**} is an iterate of k^* and β, then k^{**} can be isotoped to lie in $N(k \cup \tau)$. By isotoping k^{**}, if necessary, we have that $k^{**} \cap N(\tau)$ consists of a collection of arcs parallel to τ.

Theorem 3.2. *For each pair of integers $g \geq 1$ and $n \geq 1$, there are tunnel number one knots K such that there is an essential meridional surface $\hat{S}$ in the exterior of K, of genus g, and with $2n$ boundary components. Furthermore, $\hat{S}$ is meridionally incompressible.*

Proof. Let k be a tunnel number one knot. Suppose that k has an unknotting tunnel $\tau' = \tau_1 \cup \tau_2$, where τ_1 is a simple closed curve, and τ_2 is an arc connecting k and τ_1. Suppose there is a closed surface S of genus $g \geq 1$ embedded in the exterior of k, which is special with respect to k and τ'.

S divides S^3 into two parts, M_1 and M_2, where, say, τ_1 is contained in M_1. $M_2 \cap N(\tau')$ is a cylinder $R \cong D^2 \times I$, so that $R \cap S$ is a disk D_1, and $R \cap N(k)$ is a disk D_0. Slide τ_1 over τ_2, to get an arc τ with both endpoints on $D_0 \subset N(k)$, so that $\tau \cap M_2$ consists of two straight arcs contained in R. The surface S and the arc τ intersect in two points.

Let k^* be an iterate of k and τ; then $k^* = k_1 \cup k_2$, where k_2 is an arc parallel to τ, so it intersects S in two points. Now k_1 is an arc in $\partial N(k)$ whose endpoints lie on D_0. By pushing k_1 into the interior of $N(k)$ we get a properly embedded arc in $N(k)$. Clearly k_1 can be chosen so that $(N(k), D_0, k_1)$ forms a good tangle, just by taking an arc whose wrapping number in $N(k)$ is ≥ 2. Note that k_1 has no local knots in $N(k)$, for it is parallel to an arc lying in $\partial N(k)$. If k is parallel to a curve on S, let λ be the curve on $\partial N(k)$ which cobounds an annulus with a curve on S, so that λ meets D_0 in one arc; let $A = \eta(\gamma \cup D_0)$. Clearly k_1 can be chosen so that $(N(k), D_0, k_1)$ is good with respect to A, say by twisting k_1 meridionally as many times as necessary; this can be done because the annulus A goes longitudinally once around $N(k)$, and the wrapping number of k_1 in $N(k)$ is ≥ 2. So k^* can be chosen to be specially knotted in $N(k \cup \tau)$. It follows from Theorem 2.1 that $\hat{S} = S \cap E(k^*)$ is an essential meridional surface in $E(k^*)$, and $\partial \hat{S}$ consists of two meridians of k^*.

This implies that k^* is a tunnel number one knot which has an unknotting tunnel $\beta' = k \cup \delta$, where δ is a straight arc connecting k^* and k. Let β be the arc obtained after sliding k over δ. $N(k^* \cup \beta)$ can be chosen so that it is contained in $N(k \cup \tau)$. Let k^{**} be an iterate of k^* and β, so $k^{**} = \kappa_1 \cup \kappa_2$, where $\kappa_1 \subset \partial N(k^*)$, and κ_2 is the tunnel β. Note that $k^{**} \subset N(k \cup \tau)$. The arc κ_1 can be isotoped so that $\kappa_1 \cap N(\tau)$ consist of straight arcs, and it can be chosen so that $\kappa_1 \cap N(\tau)$ consists of n arcs, n being a fixed positive integer. So k^{**} intersects S in $2n$ points. $k^{**} \cap N(k)$ then consists of n arcs, $\rho_0, \dots, \rho_n$, which are properly embedded on $N(k)$, and whose endpoints lie in D_0. Clearly k^{**} can be chosen so that $(N(k), D_0, \rho)$ forms a good tangle, say by choosing them so that each arc ρ_i, except one, is parallel to k_1, and so that each has wrapping number ≥ 2. The remaining arc can be chosen to be a band sum of the arc k_1 and the knot k, so it can be chosen to have wrapping number ≥ 3. If k is parallel to S, again k^{**} can be chosen so that $(N(k), D_0, \rho)$ is good with respect to A. Then by Theorem 2.1, the surface $\hat{S} = S \cap E(k^{**})$ is an essential meridional surface in $E(k^{**})$, and $\partial \hat{S}$ consists of $2n$ meridians of k^{**}.

If S is meridionally incompressible, then $\hat{S}$ is meridionally incompressible. If S is meridionally compressible, then k^* and k^{**} can be chosen so that $\hat{S}$ is meridionally incompressible. $\square$

It follows from the proof of Theorem 3.2 that for the knots k constructed in [E2], there are many iterates of k, whose exteriors contain an essential meridional surface. This is because for such knots, there is an unknotting tunnel τ' and a surface S which is special with respect to k and τ'. Note also that some of these knots k are parallel to the surface S, while others are not [E2,8.2].

An example which illustrates Theorem 3.2 is shown in Figure 4. Let k be the (2,-11)-cable of the left hand trefoil; there is a torus S and unknotting tunnel τ' for k, so that S is special with respect to k and τ'. Note that k is parallel to a curve on S. The knot k^* shown in Figure 4 is an iterate of k and the tunnel τ'. It is not difficult to check that

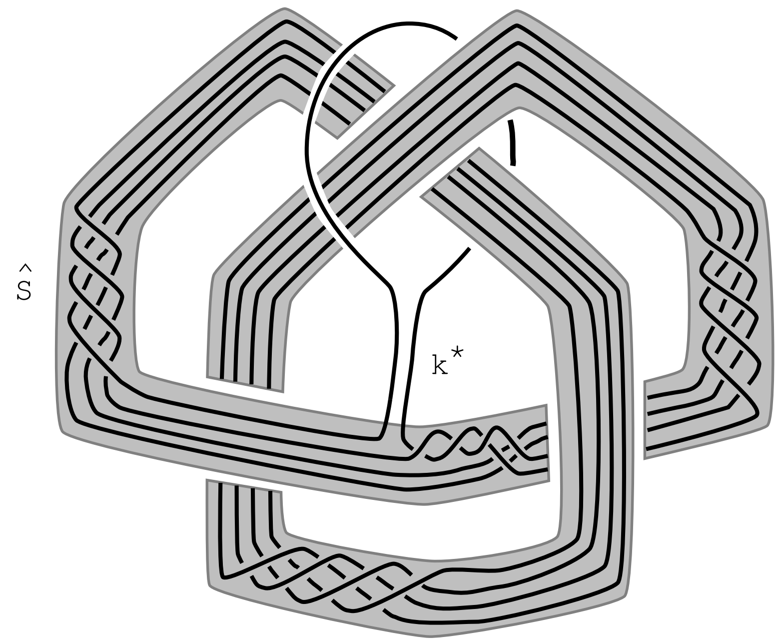

FIGURE 4

k^* satisfies the conditions of Theorem 2.1. So it follows that $\hat{S}$ is an essential meridional surface in $E(k^*)$.

Combining the last theorem and the construction given in [E2,§6], we get the following.

Theorem 3.3. *For each positive integer n, there are tunnel number one knots K, such that in the exterior of K there are n disjoint, non-parallel, closed incompressible surfaces. Each of the surfaces has genus n. One of the surfaces is meridionally compressible; the others are meridionally incompressible.*

Proof. Recall the construction given in [E2,§6]. Let k_1 be a knot, $\tau' = \tau_1 \cup \tau_2$ an unknotting tunnel, and S_1 an essential surface of genus g embedded in $E(k_1)$, which intersects τ' in one point. So S_1 is special with respect to k_1 and τ' (in both definitions, the one given here and the one in [E2,§6]; see [E2,6.1], which shows that this is true). Let $T = \partial N(k_1)$. Let A be an annulus contained in T, and let α be the core of this annulus. Suppose that α wraps around $N(k_1)$ at least twice longitudinally. If k_1 is parallel to S_1, suppose also that $\Delta(\gamma, \alpha) \geq 2$, where γ is a curve on $\partial N(k_1)$ which cobounds an annulus with a curve on S_1.

S_1 divides S^3 into two parts, M_1 and M_2, where, say, k_1 is contained in M_2. Let $\tau_2' = M_2 \cap \tau_2$; so τ_2' is an arc with an endpoint on S_1 and the other on $\partial N(k)$, which we assume lies on the curve α. The curve α goes around $N(k)$ at least twice longitudinally, then it is a toroidal graph of type 1 in $N(k)$, as defined in [E2,§4]. Let $M = M_2 - int\, N(k)$. M is a 3-manifold with incompressible boundary. To show that $\tau_2' \cup \alpha$ is a cabled graph in M_2, as defined in [E2,§6], it suffices to prove that S_1 remains incompressible after Dehn filling M along $\partial N(k)$ with slope α. If k is not parallel to a curve on S_1, then as $\Delta(\alpha, \mu) \geq 2$, this follows from the main Theorem of [Wu]. If k is parallel to a curve

on S_1, then by hypothesis, $\Delta(\alpha, \gamma) \geq 2$, and by [CGLS,2.4.3] it follows that S_1 remains incompressible.

$N(\tau_2')$ is a cylinder $R \cong D^2 \times I$, so that $R \cap S_1$ is a disk D_1, and $R \cap N(k_1)$ is a disk D_0. Assume that $D_0 \subset A$. Consider the manifold $W = M_1 \cup R \cup N(A)$, and let $\Sigma = \partial W$. As $\tau_2' \cup \alpha$ is a cabled graph in M_2, it follows from [E2,6.3] that Σ is incompressible in $S^3 - int\, W$.

Let τ be the arc obtained by sliding τ_1 over τ_2, so that $M_2 \cap \tau \subset R$. Now take an iterate k_2 of k_1 and τ of a special form. As before $k_2 = \kappa_1 \cup \kappa_2$, where $\kappa_2 = \tau$, and κ_1 is an arc in $\partial N(k_1)$. Suppose that κ_1 is contained in A, so that its wrapping number in $N(A)$ is ≥ 2 (i.e., $\rho = k_2 \cap N(A)$ is a properly embedded arc in $N(A)$ whose endpoints lie on $D_0' = R \cap \partial N(A)$, and we are requiring that the curve obtained from ρ by joining its endpoints with an arc lying on D_0' has wrapping number ≥ 2 in $N(A)$). Then $k_2 \subset W$, and it follows from [E2,6.4] that Σ is incompressible and meridionally incompressible in (W, k_2). So Σ is a meridionally incompressible surface contained in the exterior of k_2 of genus $g+1$. By [E2,8.2], it follows that k_2 is not parallel to a curve lying on Σ

It is not difficult to see that the knot k_2 also satisfies the hypothesis of Theorem 2.1; in particular, note that the arc κ_1 has wrapping number ≥ 4 in $N(k_1)$ (for κ_1 has wrapping number ≥ 2 in $N(A)$, and α has winding number ≥ 2 in $N(k_1)$). Therefore the surface $\hat{S} = S_1 \cap E(k_2)$ is meridionally incompressible in $E(k_2)$, its boundary consists of two meridians of k_2. By tubing $\hat{S}$, we get two closed surfaces in $E(k_2)$, of genus $g+1$. By an application of the handle addition Lemma [J], one of the surfaces must be incompressible in $E(k_2)$; this has to be the surface lying on M_2, for the one lying in M_1 bounds a handlebody. Denote by $\bar{S}$ such an incompressible surface; note that it is meridionally compressible. Then there are two different closed incompressible surfaces in $E(k_2)$, Σ and $\bar{S}$. By isotoping $\bar{S}$ into W, these surfaces become disjoint and are obviously non-parallel.

Note that there is an unknotting tunnel $\beta' = k_1 \cup \delta$ for k_2, where δ is a straight arc in $N(k_1)$ connecting k_1 and k_2 which intersects both surfaces Σ and $\bar{S}$ in one point. Then Σ and $\bar{S}$ are both special with respect to k_2 and β. Note that $\bar{S}$ is closer to k_2 and Σ is closer to k_1; that is, the arc δ, when going from k_2 to k_1, intersects first $\bar{S}$ and then Σ.

We have proved that there is a tunnel number one knot k_2 which has an unknotting tunnel $\tau' = \tau_1 \cup \tau_2$, and two disjoint, non-parallel closed incompressible surfaces in its exterior, each of genus $g+1$, denoted by Σ and $\bar{S}$, and which are special with respect to k_2 and τ'. Σ is meridionally incompressible and $\bar{S}$ is meridionally compressible, and the arc τ_2, when going from k_2 to τ_1 intersects first $\bar{S}$ and then Σ. Furthermore, k_2 is not parallel to a curve lying on any of the two surfaces.

Suppose by induction that we have a tunnel number one knot k_n, which has an unknotting tunnel $\tau' = \tau_1 \cup \tau_2$, and n disjoint, non-parallel closed incompressible surfaces in its exterior, of genus $g+n$, denoted by $S_1, S_2, \ldots, S_n$, which are special with respect to k_n and τ'. $S_2, \ldots, S_n$ are meridionally incompressible and S_1 is meridionally compressible, and the arc τ_2, when going from k_n to τ_1 intersects the surfaces in the order $S_1, S_2, \ldots, S_n$. Furthermore, k_n is not parallel to a curve lying on any of the surfaces.

The above construction can be repeated with k_n, $\tau' = \tau_1 \cup \tau_2$ and $S_1, S_2, \ldots, S_n$. S_i divides S^3 into M_1^i and M_2^i, where k_n lies in M_2^i. Clearly, if $i < j$ then $M_2^i \subset M_2^j$. Let α be

a simple closed curve on $\partial N(k_n)$, which goes at least twice longitudinally around $N(k_n)$. Suppose that the endpoint of τ_2 lies on α. Let $\tau_2^i = M_2^i \cap \tau_2$; then, as above, $\alpha \cup \tau_2^i$ is a cabled graph in M_2^i, for k_n is not parallel to a curve lying on S_i.

Let R_i be a regular neighborhood of τ_2^i in M_2^i, so that $R_i \cap S_i$ is a disk D_1^i, and $R \cap N(k_n)$ is a disk D_0^i. Assume that $D_0^i \subset A$, where $A = \eta(\alpha)$. Consider the manifold $W_i = M_1^i \cup R_i \cup N_i(A)$, where $N_i(A)$ is a neighborhood of A. Let $\Sigma_i = \partial W_i$. As $\tau_2^i \cup \alpha$ is a cabled graph in M_2^i, it follows from [E2,6.3] that Σ_i is incompressible in $S^3 - int\, W_i$. The neighborhoods $R_i \cup N_i(A)$ can be chosen to be thinner if $j > i$, that is, $M_2^i \cap (R_j \cup N_j(A)) \subset R_i \cup N_i(A)$ if $i < j$. Then the surfaces $\Sigma_1, \Sigma_2, \dots, \Sigma_n$ are disjoint.

Let τ be the arc obtained by sliding τ_1 over τ_2, so that $M_2^i \cap \tau \subset R_i$, for all i. Now take an iterate k_{n+1} of k_n of a special form. As before $k_{n+1} = \kappa_1 \cup \kappa_2$, where $\kappa_2 = \tau$, and κ_1 is an arc in $\partial N(k_n)$. Suppose that κ_1 is contained in A, so that its wrapping number in $N_n(A)$ is ≥ 2. Then $k_{n+1} \subset W_i$, and it follows from [E2,6.4] that Σ_i is incompressible and meridionally incompressible in (W_i, k_{n+1}). So Σ_i is a meridionally incompressible surface in the exterior of k_{n+1} of genus $g + n + 1$. Again by [E2,8.2], it follows that k_{n+1} is not parallel to a curve lying on Σ_i

The knot k_{n+1} intersects the surface S_n in two points, the wrapping number of κ_2 in $N(k_n)$ is ≥ 4, and k_n is not parallel to a curve on S_n. So k_{n+1} and S_n satisfy the conditions of Theorem 2.1, and then $\hat{S}_n = S_n \cap E(k_{n+1})$ in an incompressible, meridionally incompressible surface in $E(k_{n+1})$ whose boundary consists of two meridians of $E(k_{n+1})$. Then as above, by tubing $\hat{S}_n$ on the side of M_2^n and isotoping into W_n, we get a closed surface Σ_{n+1} which is incompressible but meridionally compressible in the exterior of k_{n+1}. The tube added to the surface can be chosen so that it lies in the interior of $R_n \cup N_n(A)$; this ensures that Σ_{n+1} is disjoint from Σ_i, for $1 \leq i \leq n$.

(From the surface S_i we can also get a meridionally compressible surface Σ_i', but it will intersect Σ_j, if $i < j$. But note that $\Sigma_{n+1} = \Sigma_n', \Sigma_{n-1}', \dots, \Sigma_2', \Sigma_1', \Sigma_1$, are disjoint).

There is an unknotting tunnel β' for k_{n+1} of the form $\beta' = k_n \cup \delta$, where δ is a straight arc in $N(k_n)$ connecting k_n and k_{n+1}. Note that δ intersects each surface Σ_i in one point; this implies that Σ_i is special w.r.t. k_{n+1} and β'. Note also that the arc δ, when going from k_{n+1} to k_n, intersects the surfaces in the order $\Sigma_{n+1}, \Sigma_n, \dots, \Sigma_1$. Finally, note that the surfaces cannot be parallel, for if two of them were, then two of the surfaces S_i would also be parallel.

This shows that k_{n+1}, β', and $\Sigma_{n+1}, \Sigma_n, \dots, \Sigma_1$ satisfy the induction hypothesis. This completes the proof.

By starting with a surface S of genus 1, and repeating the construction $n-1$ times, we get the desired conclusion. $\square$

Remark. It follows from the proof of the above theorem that by changing the induction hypothesis, we can find a tunnel number one knot k, with n incompressible surfaces in its exterior, $S_1, S_2, \dots, S_n$, so that S_n is meridionally incompressible, but S_i, for $1 \leq i \leq n-1$, is meridionally compressible, and S_n is the surface which is farthest from the knot. It follows also that there are tunnel number one knots whose exteriors contain two collection of disjoint incompressible surfaces, $S_1, \dots, S_n$, and $\Sigma_1, \dots, \Sigma_n$, where the S_i are meridionally incompressible, and the Σ_i are meridionally compressible.

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